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NEW METHOD IN THE THEORY OF LIGHT SCATTERING

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A tentative method for simplifying the theory of light scattering, based on the probability of light quantum yield from a given site of the medium in a given direction, is developed. The formula is applicable for calculating the energy scattered by the medium in different directions, for any position of the light source. Application examples for the formula include determination of the brightness distribution over a stellar disk, emission intensity of a stellar atmosphere, intensity distribution within a spectral line, and luminosity of a body of arbitrary shape. Generalization to a layer of finite optical thickness and arbitrary scattering indicatrix is given.

The theory of light scattering in stellar and planetary atmospheres has made great progress in recent years. The scope of the problems under question has greatly expanded, and a number of effective solution methods have been worked out. However, many important problems still remain to be solved. Of the existing methods of solution some are extremely awkward, others are insufficiently accurate, and still others have a limited area of application. Therefore, a search for new methods in the theory of light scattering is quite desirable.

The present article points out the advisability of introducing a new concept into the theory of light scattering, namely, the probability of light quantum yield from a given site of the medium in a determined direction. The introduction of this concept greatly simplifies the solution of various problems of light scattering, and the physical meaning of the solution itself becomes more intelligible.

I.

Let us first examine the problem of the diffuse reflection of light from a plane layer of infinitely large optical thickness. We will assume that, in the elementary act of scattering, the probability of "survival" of the quantum is equal to λ (i.e., the ratio of the scattering coefficient to the true absorption coefficient is equal to $\frac{\lambda}{1 - \lambda}$), and the probability of scattering to different sides is identical (i.e., the scattering indicatrix is spherical). Let the layer be illuminated by parallel rays of an intensity equal to I_0 and let the angle of incidence be ϑ_0. Wanted is the intensity of radiation diffusely reflected from the layer in different directions.

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* Numbers given in the margin indicate pagination in the original foreign text.

Usually, the above problem reduces to the simultaneous solution of the radiative transport equation

$$\cos\vartheta \frac{dI(\tau,\vartheta,\vartheta_0)}{d\tau} = I(\tau,\vartheta,\vartheta_0) - B(\tau,\vartheta_0) \tag{1}$$

and radiation balance equation

$$B(\tau,\vartheta_0) = \frac{\lambda}{2}\int_0^{\pi} I(\tau,\vartheta,\vartheta_0)\sin\vartheta\, d\vartheta + \frac{\lambda}{4\pi} I_0 e^{-\tau\sec\vartheta_0}. \tag{2}$$

In these equations, $I(\tau, \vartheta, \vartheta_0)$ is the intensity of radiation at an optical depth τ at an angle ϑ to the external normal and $B(\tau, \vartheta_0)$ is the ratio of the radiation coefficient to the absorption coefficient at an optical depth τ.

From eqs.(1) and (2) we easily derive one integral equation determining /356 the quantity $B(\tau, \vartheta_0)$:

$$B(\tau,\vartheta_0) = \frac{\lambda}{2}\int_0^{\infty} E_1|\tau-\tau'|B(\tau',\vartheta_0)\,d\tau' + \frac{\lambda}{4\pi} I_0 e^{-\tau\sec\vartheta_0}, \tag{3}$$

where $E_1 x = \int_x^{\infty} e^{-x}\frac{dx}{x}$. If the quantity $B(\tau, \vartheta_0)$ is known, then the intensity of radiation emerging from the layer of interest here is found by the formula

$$I(0,\vartheta,\vartheta_0) = \int_0^{\infty} B(\tau,\vartheta_0)e^{-\tau\sec\vartheta}\sec\vartheta\, d\tau. \tag{4}$$

Let us now solve the same problem by a different method. We will designate in terms of $p(\tau, \vartheta)d\omega$ the probability that the light quantum absorbed at optical depths τ emerges from the medium at an angle ϑ to the normal within the solid angle $d\omega$. It is easy to set up the integral equation for determining the quantity $p(\tau, \vartheta)$.

The quantity $p(\tau, \vartheta)$ is made up of two parts: the probability of quantum yield from the medium without scattering along the paths and the probability of quantum yield from the medium after multiple scattering. It is obvious that the first part will be equal to $\frac{\lambda}{4\pi} e^{-\tau\sec\vartheta}$. To find the second part we must multiply the probability that the quantum absorbed at a depth τ will then be absorbed at a depth τ' by the probability of quantum yield from the medium from a depth τ' and integrate this product with respect to τ' from 0 to ∞. In other words, the second part will be equal to

$$\int_0^{\infty} p(\tau',\vartheta)\,d\tau' \frac{\lambda}{4\pi} 2\pi \int_0^{\frac{\pi}{2}} e^{-|\tau-\tau'|\sec\vartheta'}\sec\vartheta'\sin\vartheta\, d\vartheta'.$$

However, the inner integral in this expression is equal to $E_1|\tau - \tau'|$. Therefore, the quantity $p(\tau, \vartheta)$ will be determined by the following integral equation:

$$p(\tau, \vartheta) = \frac{\lambda}{4\pi} e^{-\tau \sec \vartheta} + \frac{\lambda}{2} \int_0^\infty E_i |\tau - \tau'| \, p(\tau', \vartheta) \, d\tau'. \tag{5}$$

If the probability of quantum yield from the medium from a given depth is known and if we also know the quantity of energy arriving from the light source and being absorbed at this depth, then by multiplying these quantities we will obviously obtain the quantity of energy emerging from the medium from the depth in question. Integration of this product with respect to all depths will give the total energy issuing from the medium (in a determined direction). In the examined case (i.e., when a plane layer is illuminated by parallel rays of intensity I_0 at an angle of incidence of ϑ_0) a quantity of energy equal to $I_0 e^{-\tau \sec \vartheta_0} d\tau$ is absorbed at the optical depth τ in an elementary volume with a cross section of 1 cm^2 and an optical thickness of $d\tau$. Multiplying this quantity by $p(\tau, \vartheta)$ and integrating with respect to τ from 0 to ∞, we find the energy escaping from the medium through 1 cm^2 of the surface at an angle ϑ to the normal within unit solid angle. The unknown intensity of radiation emerging /357 from the medium will be equal to

$$I(0, \vartheta, \vartheta_0) = \int_0^\infty p(\tau, \vartheta) I_0 e^{-\tau \sec \vartheta_0} \sec \vartheta \, d\tau. \tag{6}$$

Let us compare the results obtained by the two methods. Equations (3) and (5) indicate that

$$p(\tau, \vartheta) = \frac{B(\tau, \vartheta)}{I_0}, \tag{7}$$

i.e., the probability of quantum yield from the medium $p(\tau, \vartheta)$ is equal to the function $B(\tau, \vartheta)$ already known in the theory of scattering when $I_0 = 1$. Comparing eqs.(4) and (6) and using in this case the relations (7), we also find

$$I(0, \vartheta, \vartheta_0) \cos \vartheta = I(0, \vartheta_0, \vartheta) \cos \vartheta_0. \tag{8}$$

The so-called "principle of reversibility" for optical phenomena is expressed by eq.(6).

Equations (7) and (8) derived above are, of course, interesting. However, they do not represent the essence of the matter. What is important is that, in introducing into the examination the probability of quantum yield from the medium $p(\tau, \vartheta)$ and in setting up eq.(5) which determines the quantity $p(\tau, \vartheta)$, we made no assumptions as to the mode of appearance of the light quanta in the medium. However, once we have defined the quantity $p(\tau, \vartheta)$, there will be no difficulty in calculating the energy scattered by the medium in different directions, for any position of the light sources.

Let us assume at first that the illumination produced by the light sources is identical for all sites located at one and the same depth. Let $f(\tau)d\tau$ be the quantity of energy absorbed within 1 sec by an elementary volume with a cross section of 1 cm^2 and an optical thickness $d\tau$, at an optical depth τ. It is obvious that by using the quantity $p(\tau, \vartheta)$ we can write the following expression for the intensity of radiation emerging from the medium at an angle ϑ to the normal:

$$I(\vartheta) = \int_0^\infty f(\tau)\, p(\tau, \vartheta) \sec \vartheta \, d\tau. \tag{9}$$

If the medium is illuminated by parallel rays of intensity I_0 and at an angle of incidence of ϑ_0, then

$$f(\tau) = I_0 e^{-\tau \sec \vartheta_0}, \tag{10}$$

and in this case we replace eq.(9) by the previously presented eq.(6).

Let us give other examples of using eq.(9). Let the light sources be located uniformly at an optical depth τ in a thin layer having a thickness $\Delta\tau$ and let them emit an identical quantity of energy in different directions. We will designate the quantity of energy emitted in 1 sec by an elementary volume of the layer with a cross section of 1 cm^2 in terms of $4\pi B_0 \Delta\tau$. In this case, for the function $f(\tau)$ we should obviously take the quantity $\frac{4\pi}{\lambda} B_0$. Therefore, by means of eq.(9) we find

$$I(\vartheta) = \frac{4\pi}{\lambda} B_0 \Delta\tau\, p(\tau, \vartheta) \sec \vartheta. \tag{11}$$

If the luminous layer is located at very great optical depths, then mainly diffuse radiation of the medium will reach the observer. Here the relative /358
distribution of the radiation intensity by angles will obviously be independent of the depth of the luminous layer. In other words, the indicated distribution can then be expressed by the formula

$$I(\vartheta) \sim p(\tau, \vartheta) \sec \vartheta \quad \text{for} \quad \tau \to \infty. \tag{12}$$

When $\lambda = 1$, i.e., in the case of pure scattering, the model of a medium with energy sources at an infinitely great depth corresponds particularly to a stellar photosphere. Therefore, eq.(12), at $\lambda = 1$, yields specifically the brightness distribution over the stellar disk.

If the light sources are distributed uniformly throughout the entire medium, then in place of eq.(11) we have:

$$I(\vartheta) = \frac{4\pi}{\lambda} B_0 \int_0^\infty p(\tau, \vartheta) \sec \vartheta \, d\tau. \tag{13}$$

It is easy to see that the intensity of the radiation escaping from the atmosphere of a star within the spectral line can be expressed by eq.(13). In this case, the quantity $p(\tau, \vartheta)$ will depend upon the frequency ν, making use of the quantities λ and τ:

$$\lambda = \frac{\sigma_\nu}{\sigma_\nu + \alpha}, \quad d\tau = (\sigma_\nu + \alpha)\, dz, \tag{14}$$

where σ_ν is the coefficient of scattering in the spectral line, α is the absorption coefficient in a continuous spectrum, dz is an element of depth in the atmosphere. As regards the radiation sources, it is known that the energy emitted in a line comes from the energy of the continuous spectrum. We will use $4\pi B^* \alpha dz$ for denoting the quantity of energy emitted in a continuous spectrum by an elementary volume with a cross section of 1 cm^2 and thickness dz, in 1 sec. This expression can also be presented in the form of $4\pi(1 - \lambda)B^* d\tau$ by means of eqs.(14). It is obvious that, in eq.(13), we can now substitute the quantity $(1 - \lambda)B^*$ for the quantity B_0. Furthermore, we will assume that in the case in question the quantity B^* is the intensity of radiation escaping from the stellar atmosphere in a continuous spectrum. Therefore, the ratio $\frac{I(\vartheta)}{B^*}$ will determine the contour of the spectral line. Denoting this ratio by $r_\nu(\vartheta)$, we obtain

$$r_\nu(\vartheta) = 4\pi \frac{1-\lambda}{\lambda} \int_0^\infty p(\tau, \vartheta) \sec \vartheta \, d\tau. \tag{15}$$

This indicates that quantities such as the brightness distribution over the stellar disk and the intensity distribution within the spectral line, which are of importance for astrophysics, are quite simply expressed in terms of the quantity $p(\tau, \vartheta)$ introduced above.

Let us now assume that the stipulation relative to the location of the light sources, made in deriving eq.(9), is not satisfied, meaning that the illumination is not identical at various sites at the same depth. Under such conditions, the intensity of radiation emerging from the medium will depend not only on the direction but also on the locus of escape. Therefore, for its determination we must introduce the probability of quantum yield from the medium, which accordingly generalizes the quantity $p(\tau, \vartheta)$ introduced above. However, it is obvious that the total energy scattered by the medium in a certain direction can be found in the general case by means of the quantity $p(\tau, \vartheta)$. If we denote by $F(\tau)d\tau$ the quantity of energy arriving from the light sources and /359
being absorbed between the optical thicknesses τ and $\tau + d\tau$, then the quantity of energy escaping from the medium at an angle ϑ to the normal in unit solid angle will be equal to

$$E(\vartheta) = \int_0^\infty F(\tau) p(\tau, \vartheta) \, d\tau. \tag{16}$$

Equation (16) can be applied primarily to a medium illuminated by a point source of light. Let the point source emit in different directions the same quantity of energy and let its luminance be equal to L. If the point source is located above the medium, then it is easy to obtain

$$F(\tau) = \frac{L}{2} E_1 \tau. \tag{17}$$

Substituting this expression for $F(\tau)$ into eq.(16) and taking into consideration eq.(5) determining $p(\tau, \vartheta)$, we then find

$$E(\vartheta) = \frac{L}{\lambda}\left[p(0, \vartheta) - \frac{\lambda}{4\pi}\right]. \tag{18}$$

It should be stated that the quantity $E(\vartheta)$ does not depend on the height of the light source above the medium.

If the point source of light is within the medium at an optical depth τ, then

$$F(\tau') = \frac{L}{2} E_1 |\tau' - \tau|, \tag{19}$$

and eq.(16) over eq.(5) will yield

$$E(\vartheta) = \frac{L}{\lambda}\left[p(\tau, \vartheta) - \frac{\lambda}{4\pi} e^{-\tau \sec \vartheta}\right]. \tag{20}$$

Of course, eq.(20) determines only the energy escaping from the medium after scattering (but not directly from the light source). Taking into account the physical meaning of the quantity $p(\tau, \vartheta)$, we could write this formula immediately.

Equations (19) and (20) may be useful for determining the energy scattered by a nebula in different directions upon its illumination by a star (or several stars).

Thus, the various problems of the theory of light scattering (which differ from one another by the location of the light sources) are quite simply solved by means of the function $p(\tau, \vartheta)$. This function itself, which represents the probability of quantum yield from a given depth in a given direction, simultaneously represents, as indicated above, the ratio of the radiation coefficient to the absorption coefficient [i.e., the function $B(\tau, \vartheta)$] in the problem on diffuse reflection of light by a plane layer when the layer is illuminated by parallel rays. This fact must be considered of prime significance since all results obtained in previous works for determining the function $B(\tau, \vartheta)$ can now be used also for determining the function $p(\tau, \vartheta)$. In particular, for the function $p(\tau, \vartheta)$ we can take the approximate expression for the function $B(\tau, \vartheta)$ derived from the system of equations (1) and (2) in solving them by methods conventional in astrophysics (averaging the radiation intensity by angles). If this expression for $p(\tau, \vartheta)$ is substituted into eqs.(12) and (15), we will arrive at the formulas known in astrophysics and giving the energy distribution over the stellar disk and the contour of the spectral line. On substituting the indi- /360
cated expression for $p(\tau, \vartheta)$ into eqs.(9) and (10) we can obtain, at proper selection of the functions $f(\tau)$ and $F(\tau)$, an approximate solution for any other problem concerning the luminosity of a plane layer.

II.

As an example of employing the above formulas, we will reproduce certain results obtained earlier by V.A.Ambartsumyan in his well-known work on the theory of light scattering.

First, let us derive a new equation for determining the function $p(\tau, \vartheta)$. Wanted is the probability of quantum yield from an optical depth $\tau + \Delta\tau$, i.e., $p(\tau + \Delta\tau, \vartheta)$. For this, let us imagine that the quantum emerges from an optical depth τ and then passes through an additional layer of thickness $\Delta\tau$. Then, for $p(\tau + \Delta\tau, \vartheta)$ we derive the following expression:

$$p(\tau + \Delta\tau, \vartheta) = p(\tau, \vartheta)(1 - \Delta\tau \sec \vartheta) + \\ + 2\pi \int_0^{\frac{\pi}{2}} p(\tau, \vartheta') \sin \vartheta' d\vartheta' \Delta\tau \sec \vartheta \, p(0, \vartheta). \tag{21}$$

The first term on the right-hand side of this relation takes into account the fact that the quantum may escape from a depth τ at an angle ϑ to the normal and pass through an additional layer without absorption, while the second term takes into account the fact that the quantum may emerge from a depth τ in any direction, may then be absorbed in the additional layer, and scattered by it in a given direction. From eq.(21) we obtain

$$\frac{\partial p(\tau, \vartheta)}{\partial \tau} = - p(\tau, \vartheta) \sec \vartheta + 2\pi p(0, \vartheta) \int_0^{\frac{\pi}{2}} p(\tau, \vartheta') \sec \vartheta' \sin \vartheta' d\vartheta'. \tag{22}$$

For further convenience we will denote $\cos \vartheta$ by η and $p(\tau, \vartheta)$ by $p(\tau, \eta)$. Then the equation determining $p(\tau, \eta)$ can be written in the form

$$\frac{\partial p(\tau, \eta)}{\partial \tau} = - \frac{1}{\eta} p(\tau, \eta) + 2\pi p(0, \eta) \int_0^1 p(\tau, \eta') \frac{d\eta'}{\eta'}. \tag{23}$$

An analogous equation was previously derived by V.A.Ambartsumyan for determining the function $B(\tau, \eta)$ directly from the integral equation (3) and used by him for solving the problem of diffuse reflection of light by a plane layer [eq.(1)]. Then, using eq.(23) other problems concerning the luminosity of the layer can be solved.

We will first present the solution of the problem of diffuse reflection of light from the layer. Let us denote $\cos \vartheta_0$ by ξ and, in place of the intensity of the diffusely reflected light $I(0, \eta, \xi)$, let us introduce the brightness coefficient $\rho(\eta, \xi)$ by means of the relation

$$I(0, \eta, \xi) = \frac{I_0}{\pi} \rho(\eta, \xi) \xi. \tag{24}$$

Taking into consideration that the quantity $I(0, \eta, \xi)$ is determined by eq.(3), we then multiply both sides of eq.(23) by $e^{-\frac{\tau}{\xi}} \frac{d\tau}{\eta}$ and integrate from 0 to ∞. As a result we find

$$\rho(\eta, \xi)(\eta + \xi) = \pi p(0, \eta) \left[1 + 2\xi \int_0^1 \rho(\eta', \xi) \, d\eta'\right]. \tag{25}$$

However, it is obvious that /361

$$p(0,\eta)=\frac{\lambda}{4\pi}\left[1+2\eta\int_0^1\rho(\eta,\eta')\,d\eta'\right]. \tag{26}$$

This follows both from the physical meaning of the quantities $p(0, \eta)$ and $\rho(\eta, \eta')$ and from eq.(2) at $\tau = 0$. Therefore, designating

$$\varphi(\eta)=1+2\eta\int_0^1\rho(\eta,\eta')\,d\eta', \tag{27}$$

we derive the following expression for the brightness coefficient:

$$\rho(\eta,\xi)=\frac{\lambda}{4}\frac{\varphi(\eta)\varphi(\xi)}{\eta+\xi}, \tag{28}$$

where the function $\varphi(\eta)$ is determined from the equation

$$\varphi(\eta)=1+\frac{\lambda}{2}\eta\varphi(\eta)\int_0^1\frac{\varphi(\xi)}{\eta+\xi}\,d\xi. \tag{29}$$

Equation (29) is readily solved numerically. Its solution was obtained in final form by the author (Ref.5).

Let us now find the brightness distribution over the stellar disk. For this, using eq.(12) we must obtain the expression for $p(\tau, \eta)$ in infinitely deep layers of the medium. Assuming that, in these layers,

$$p(\tau,\eta)=C(\eta)e^{-k\tau}, \tag{30}$$

and substituting this expression for $p(\tau, \eta)$ into eq.(23), we find

$$C(\eta)=A\frac{\varphi(\eta)\eta}{1-k\eta}, \tag{31}$$

where A is an arbitrary constant and k is determined from the equation

$$\frac{\lambda}{2}\int_0^1\frac{\varphi(\eta)}{1-k\eta}\,d\eta=1. \tag{32}$$

Since we are interested in the case of pure scattering ($\lambda = 1$), and since here $k = 0$, the brightness distribution over the stellar disk will be obtained from

$$I(\eta)\sim\varphi(\eta). \tag{33}$$

Finally, let us determine the contour of the spectral line. According to eq.(15), we must then integrate the function $p(\tau, \eta)$ with respect to τ from 0 to ∞. From eq.(23) we readily obtain

$$\int_0^\infty p(\tau, \eta)\frac{d\tau}{\eta} = \frac{\lambda}{4\pi}\,\frac{\varphi(\eta)}{1 - \frac{\lambda}{2}\int_0^1 \varphi(\eta)\,d\eta}. \tag{34}$$

However, it follows from eq.(29) that /362

$$\frac{\lambda}{2}\int_0^1 \varphi(\eta)\,d\eta = 1 - \sqrt{1-\lambda}. \tag{35}$$

Therefore, substituting eq.(34) into eq.(15) and taking into account eq.(35), we find

$$r_\nu(\eta) = \varphi(\eta)\sqrt{1-\lambda}. \tag{36}$$

As already mentioned above, the results expressed by eqs.(28), (33), and (36) were obtained for the first time by V.A.Ambartsumyan in a number of papers (Ref.2, 3, 4). For this, his very elegant method involving a study of processes occurring only in the surface layer was used. Obtainment of the same results by the function $p(\tau, \eta)$ seems also of some interest.

III.

It was shown above that the introduction into the light scattering theory of a new quantity - the probability of the yield of a light quantum from a given site of a medium in a given direction - greatly facilitates the solution of various problems of the theory of light scattering. For simplicity, we considered that the medium is a plane layer of infinitely large optical thickness with a spherical indicatrix of scattering. It is easy, however, to give a generalization of the results obtained above for a layer of finite optical thickness and arbitrary scattering indicatrix. It is especially important that in the general case the probability of quantum yield from a layer will be equal to the ratio of the radiation coefficient to the absorption coefficient in the problem of the luminosity of a layer illuminated by parallel rays, i.e., the function already known in the theory of light scattering.

Problems of the luminosity of a body of arbitrary shape can also be solved by the method presented above. For this, it is necessary to determine first the probability of quantum yield from a given site of the body in a prescribed direction. Then, the quantity of energy scattered by the body in a given direction at any position of the light sources is found by integration. The problem of the luminosity of a sphere will be the next in complexity after the problem of the luminosity of a plane layer. This might be useful in determinations of the luminosity of stars and nebulae of spherical shape.

Finally, nonstationary problems of the theory of light scattering, i.e., problems of the luminosity of a body in the absence of a radiation balance, can also be solved by the proposed method. To this end, it is necessary to introduce the probability of quantum yield from a given site of the body in a prescribed direction over a certain time interval.

The author proposes to take up all problems touched upon here at a later date.

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